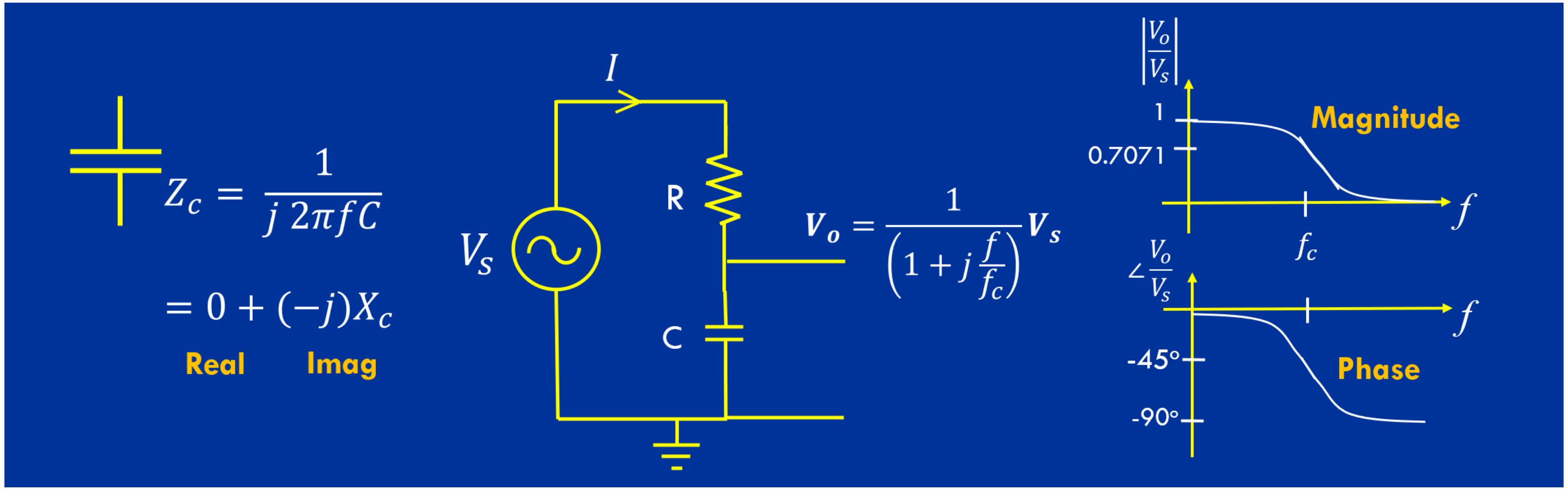
ECE Topic #6 – Complex Numbers, Phasors, Impedances, and Frequency Response

Companion video at: https://www.youtube.com/watch?v=3avBjiaAPBY

This is Episode 6 in the ECE Topics series. It covers complex numbers, phasors, and impedances - the necessary mathematical machinery for solving any "AC circuit". In keeping with other episodes on this channel, the goal is to understand circuits, and several practical examples are used as examples. While the math is covered rigorously, the pre-requisites needed are limited to algebra, and basic trig.



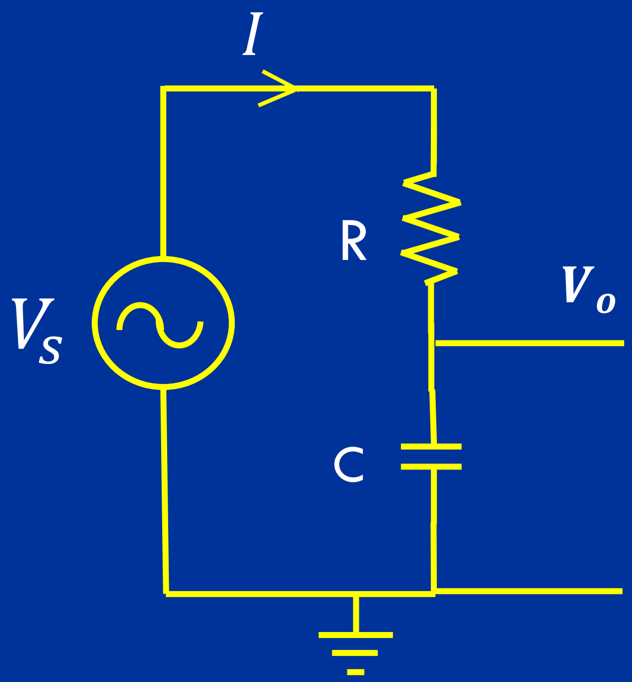
Topics in ECE #6

Complex Numbers, Phasors, Impedances and Frequency Response

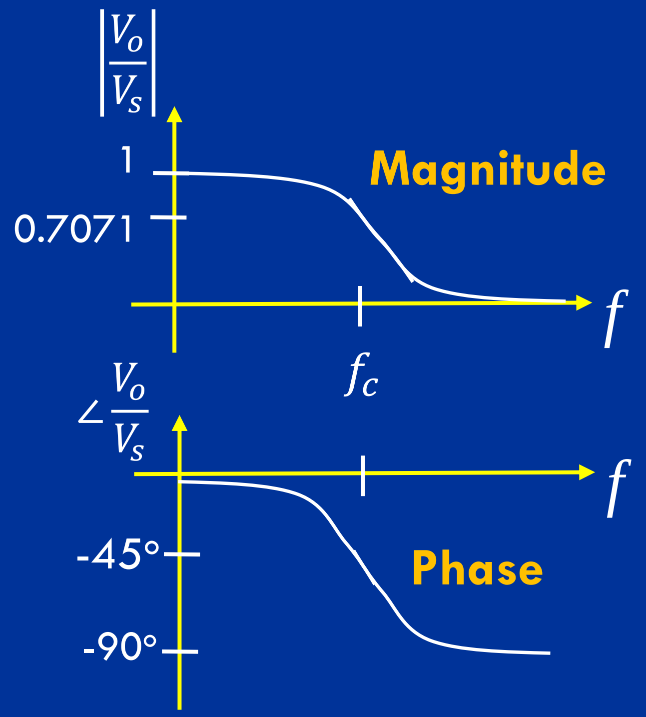
$$Z_c = \frac{1}{j\,2\pi f C}$$

$$= 0 + (-j)X_c$$

Real Imag



$$\boldsymbol{V_o} = \frac{1}{\left(1 + j\frac{f}{f_c}\right)}\boldsymbol{V_s}$$



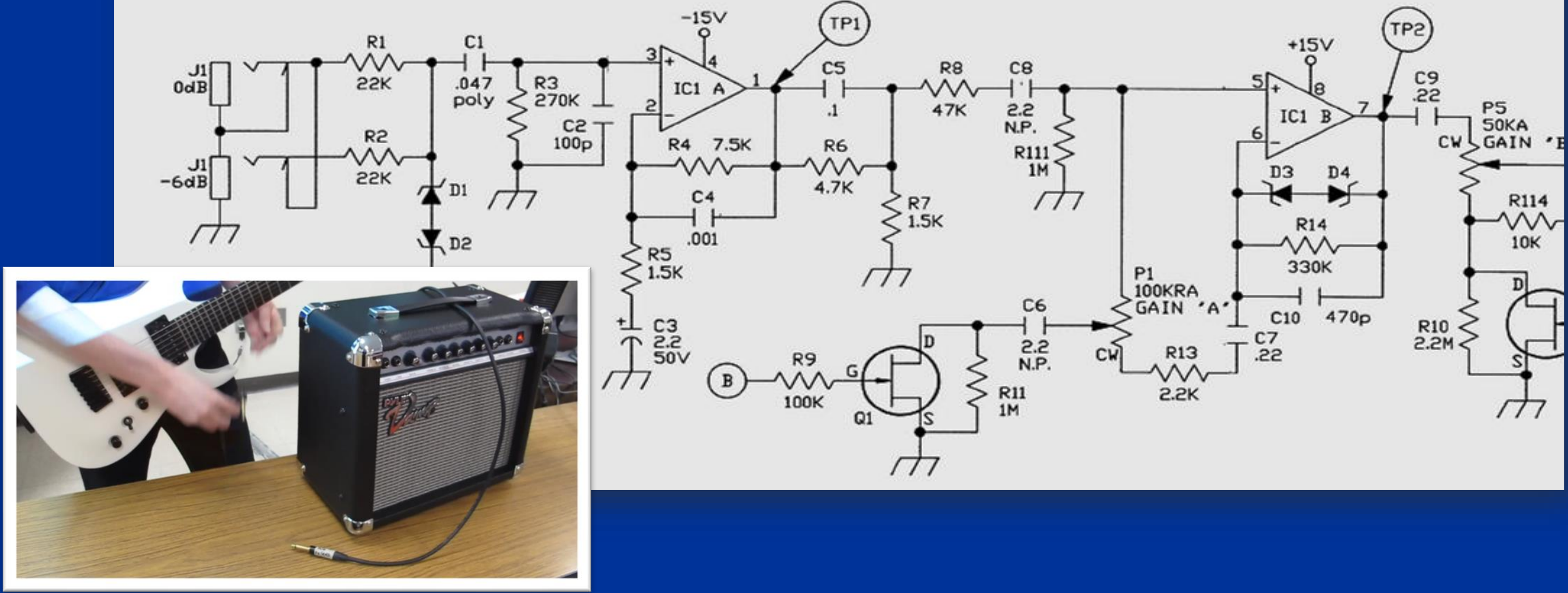
The How and Why

as well as the Math …

Application 1: Signal Processing

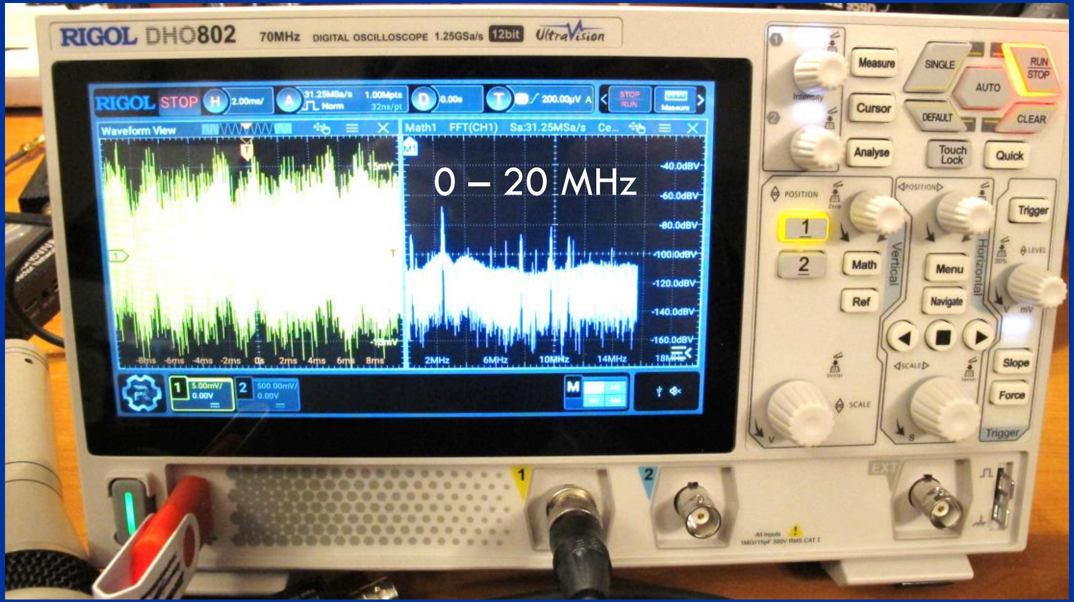
From ECE Topics - Episode 4 – Understanding Circuits

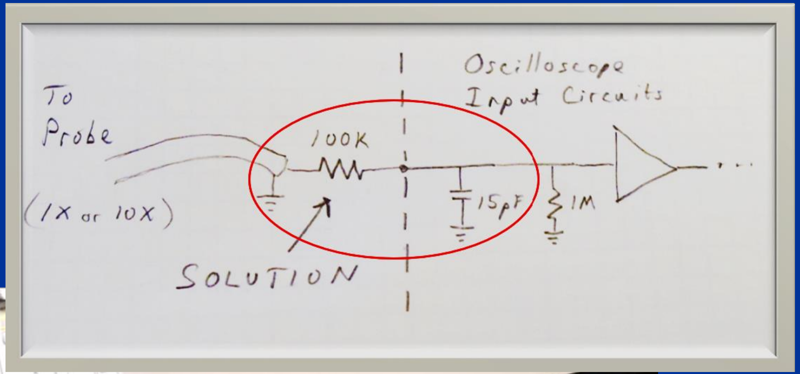
http://www.guitarnucleus.net/resources/schematics/Ampeg-AX70-Schematic.pdf



Application 2:   RFI Filtering

From ECE Topics - Episode 5 – Scopes, Mics, and Noise



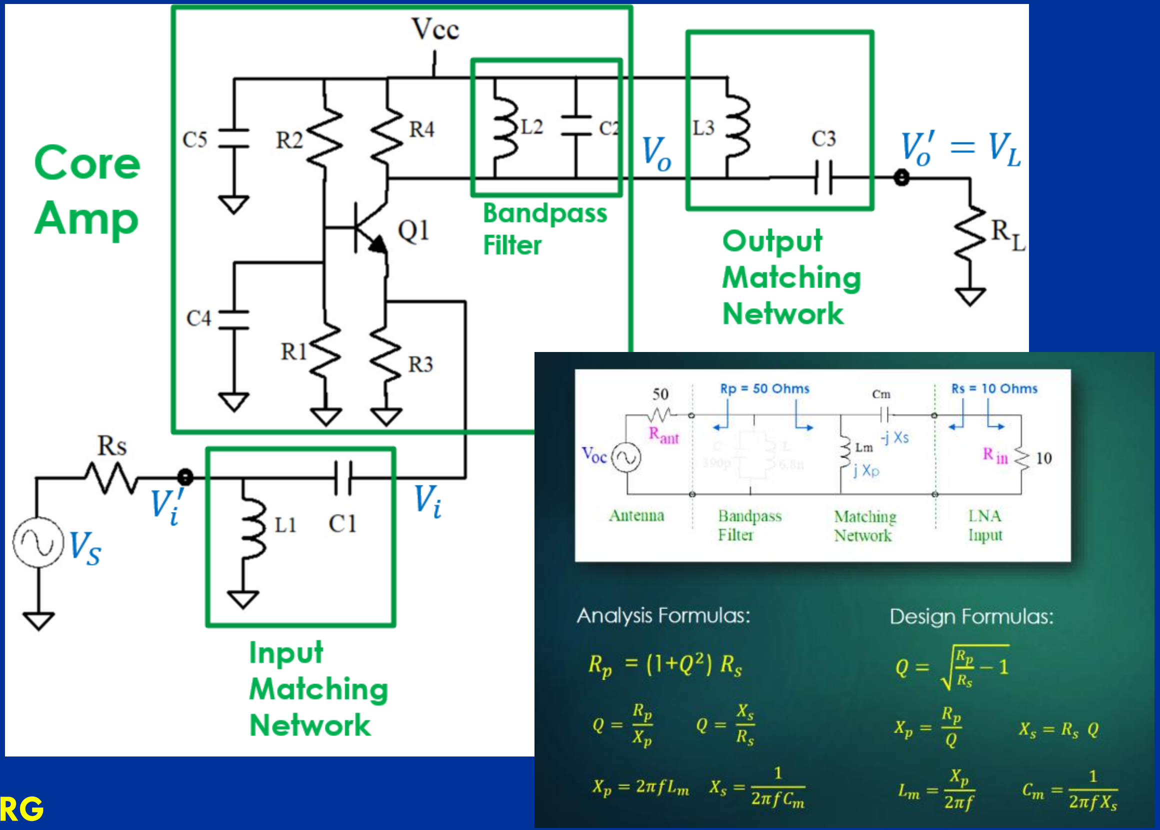


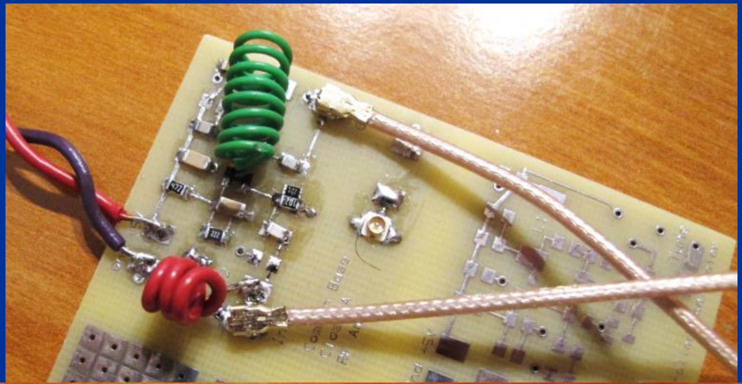
100 kHz Lowpass Filter

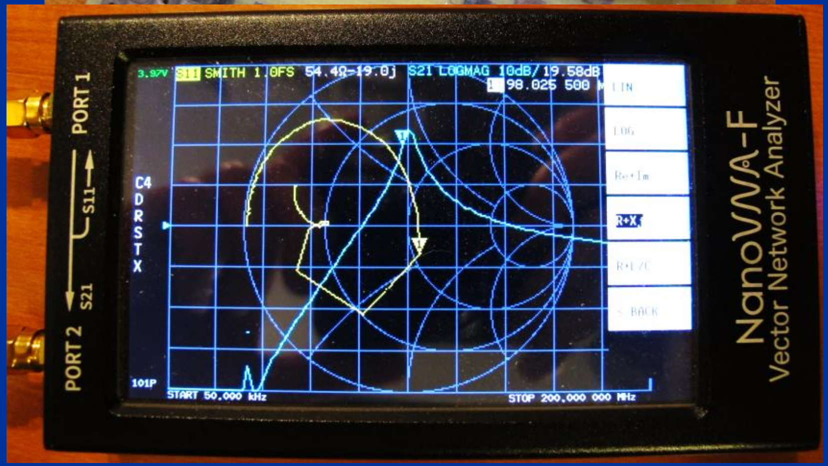
Application 3: Radio/Wireless

From Radio Design 101

Episode 2- Impedance Matching







Other Applications

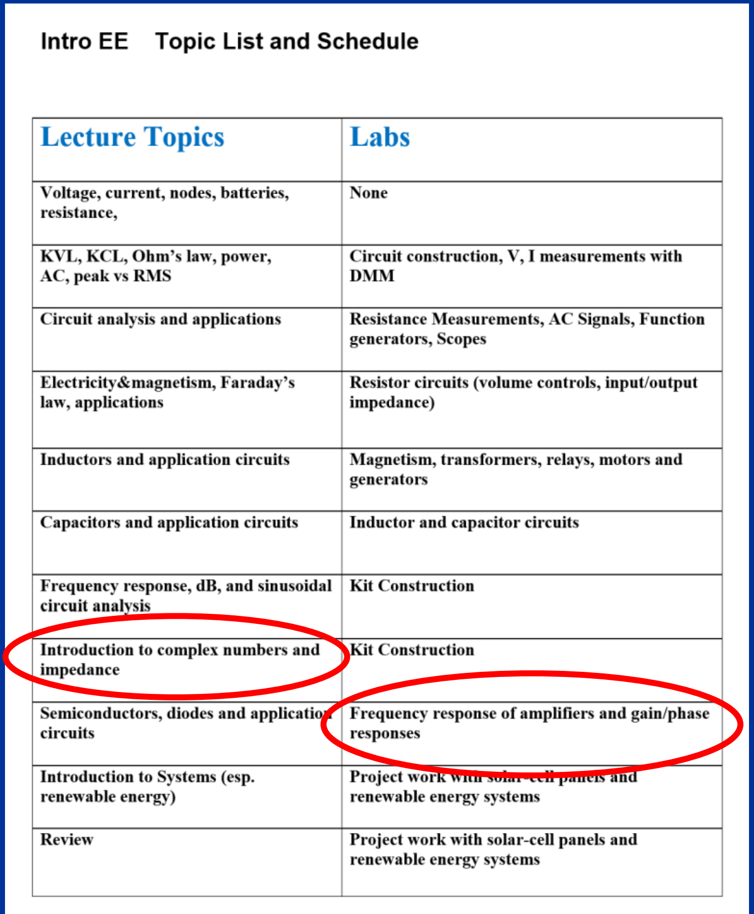
- Power systems engineering
- Any (linear) circuit analysis involving time-varying voltages and currents
- Fourier transforms, EM theory, and the physical universe in general ☺

Today's Topics

- Complex Numbers and Operations
- Phasors, Impedance, and Circuit Analysis
- Plotting Frequency Response

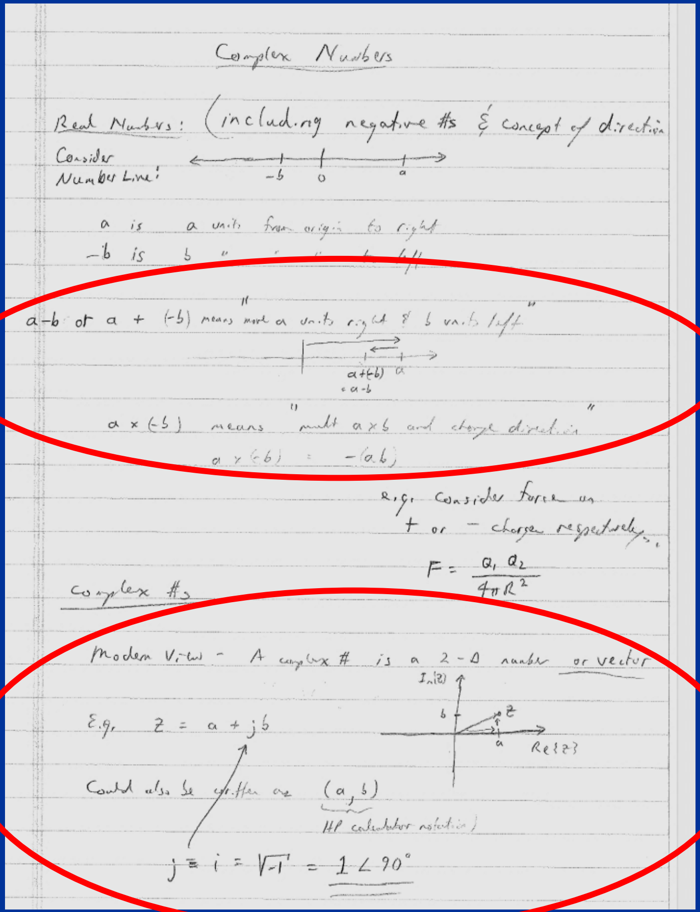
Lectures and Handouts

from Intro-EE University Course



Intro EE Topic List and Schedule

Lecture Topics	Labs
Voltage, current, nodes, batteries, resistance,	None
KVL, KCL, Ohm's law, power, AC, peak vs RMS	Circuit construction, V, I measurements with DMM
Circuit analysis and applications	Resistance Measurements, AC Signals, Function generators, Scopes
Electricity&magnetism, Faraday's law, applications	Resistor circuits (volume controls, input/output impedance)
Inductors and application circuits	Magnetism, transformers, relays, motors and generators
Capacitors and application circuits	Inductor and capacitor circuits
Frequency response, dB, and sinusoidal circuit analysis	Kit Construction
Introduction to complex numbers and impedance	Kit Construction
Semiconductors, diodes and application circuits	Frequency response of amplifiers and gain/phase responses
Introduction to Systems (esp. renewable energy)	Project work with solar-cell panels and renewable energy systems
Review	Project work with solar-cell panels and renewable energy systems



Complex Numbers

Real Numbers: (including negative #s & concept of direction

Consider Number Line: $-b$ 0 a

a is a units from origin to right

$-b$ is b " " " " left

$a-b$ or $a + (-b)$ means "move a units right & b units left"

$a+(-b) = a-b$

$a \times (-b)$ means "mult $a \times b$ and change direction"

$a \times (-b) = -(a.b)$

e.g. Consider force on + or − charges respectively,

$$F = \frac{Q_1 Q_2}{4\pi R^2}$$

Complex #s

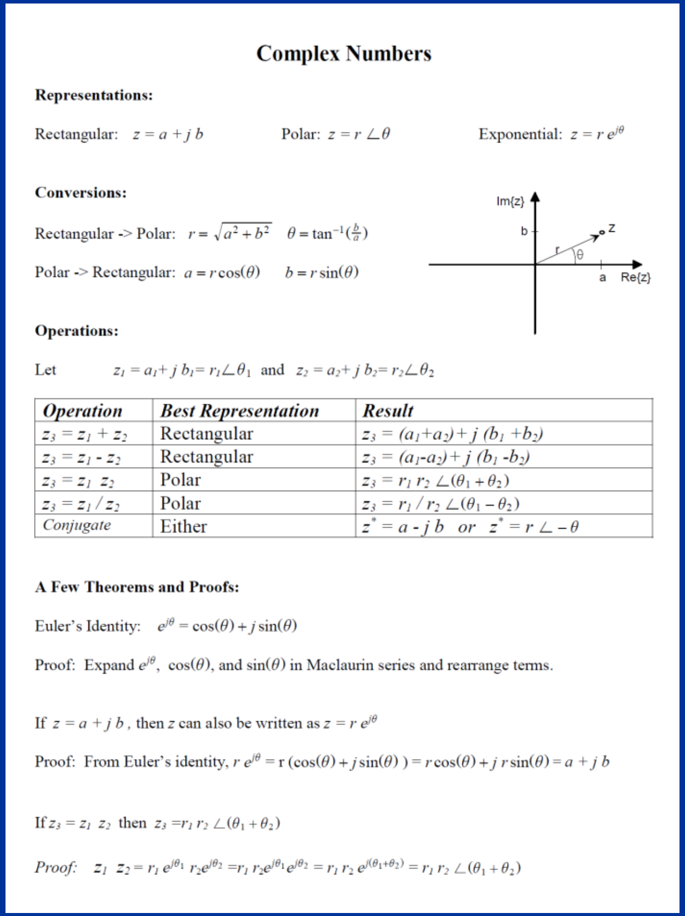
Modern View - A complex # is a 2-D number or vector

E.g. $z = a + jb$

Could also be written as (a, b)

HP calculator notation)

$j \equiv i = \sqrt{-1} = 1\angle 90°$



Complex Numbers

Representations:

Rectangular: $z = a + jb$ Polar: $z = r\angle\theta$ Exponential: $z = r\,e^{j\theta}$

Conversions:

Rectangular -> Polar: $r = \sqrt{a^2+b^2}$ $\theta = \tan^{-1}(\frac{b}{a})$

Polar -> Rectangular: $a = r\cos(\theta)$ $b = r\sin(\theta)$

Operations:

Let $z_1 = a_1 + j\,b_1 = r_1\angle\theta_1$ and $z_2 = a_2 + j\,b_2 = r_2\angle\theta_2$

Operation	*Best Representation*	*Result*
$z_3 = z_1 + z_2$	Rectangular	$z_3 = (a_1+a_2) + j\,(b_1+b_2)$
$z_3 = z_1 - z_2$	Rectangular	$z_3 = (a_1-a_2) + j\,(b_1-b_2)$
$z_3 = z_1\ z_2$	Polar	$z_3 = r_1 r_2 \angle(\theta_1+\theta_2)$
$z_3 = z_1 / z_2$	Polar	$z_3 = r_1 / r_2 \angle(\theta_1-\theta_2)$
Conjugate	Either	$z^* = a - j\,b$ or $z^* = r\angle-\theta$

A Few Theorems and Proofs:

Euler's Identity: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

Proof: Expand $e^{j\theta}$, $\cos(\theta)$, and $\sin(\theta)$ in Maclaurin series and rearrange terms.

If $z = a + j\,b$, then z can also be written as $z = r\,e^{j\theta}$

Proof: From Euler's identity, $r\,e^{j\theta} = r\,(\cos(\theta) + j\sin(\theta)) = r\cos(\theta) + j\,r\sin(\theta) = a + j\,b$

If $z_3 = z_1\ z_2$ then $z_3 = r_1 r_2 \angle(\theta_1+\theta_2)$

Proof: $z_1\ z_2 = r_1 e^{j\theta_1}\ r_2 e^{j\theta_2} = r_1 r_2 e^{j\theta_1} e^{j\theta_2} = r_1 r_2\, e^{j(\theta_1+\theta_2)} = r_1 r_2 \angle(\theta_1+\theta_2)$

Complex Numbers

IMPORTANT NOTE: The ECE field uses ***j*** instead of ***i*** for $\sqrt{-1}$

… because ***i*** is used for current ☺

Representations:

Rectangular: $z = a + j\,b$ Polar: $z = r\,\angle\theta$ Exponential: $z = r\,e^{j\theta}$

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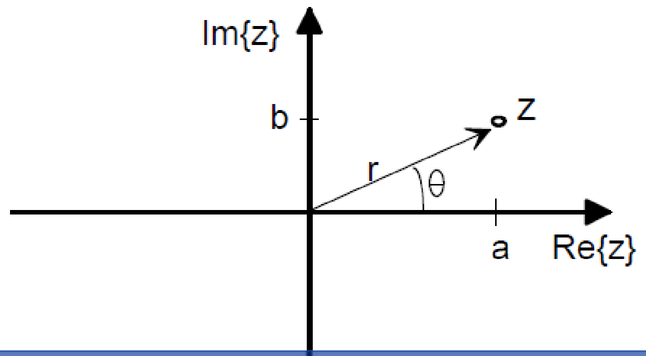
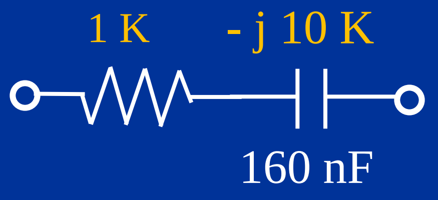


Real-world Problem Set 1

Find the impedance Z of each circuit below, to three or four significant digits.
Express each in polar form, with angle in degrees.

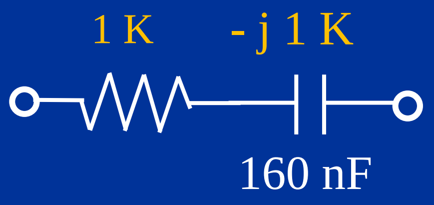
HINT: Impedances in series add

@ 100 Hz:



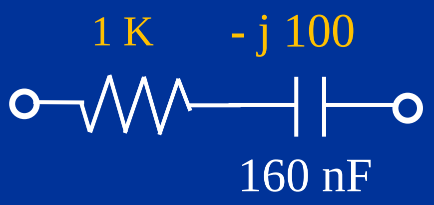
Z =

@ 1 KHz:



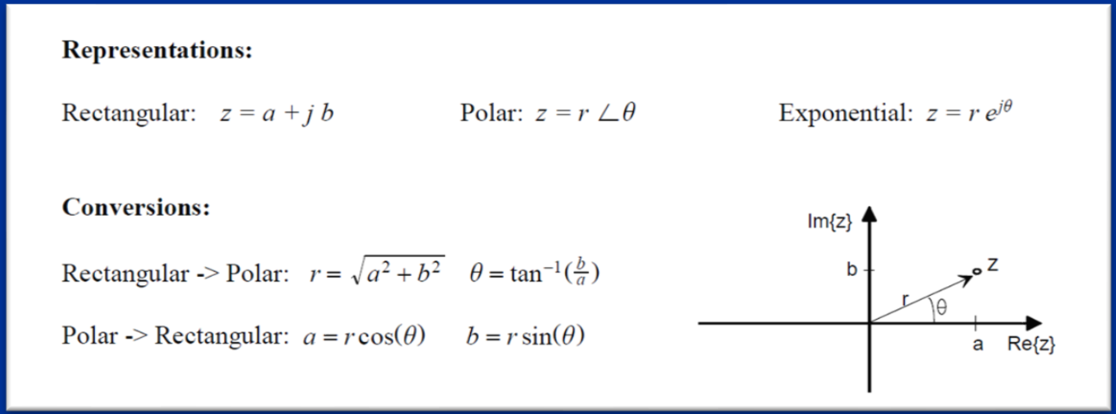
Z =

@ 10 KHz:



Z =

Problem Set 1 Solution


Representations:

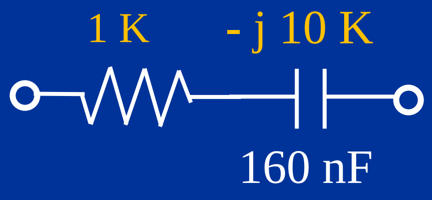
Rectangular: $z = a + j\,b$ Polar: $z = r\,\angle\theta$ Exponential: $z = r\,e^{j\theta}$

Conversions:

Rectangular -> Polar: $r = \sqrt{a^2 + b^2}$ $\theta = \tan^{-1}(\frac{b}{a})$

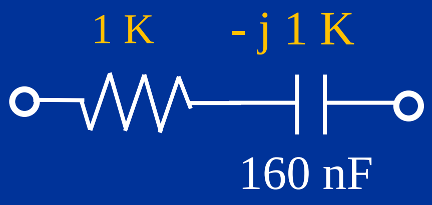
Polar -> Rectangular: $a = r\cos(\theta)$ $b = r\sin(\theta)$

@ 100 Hz:

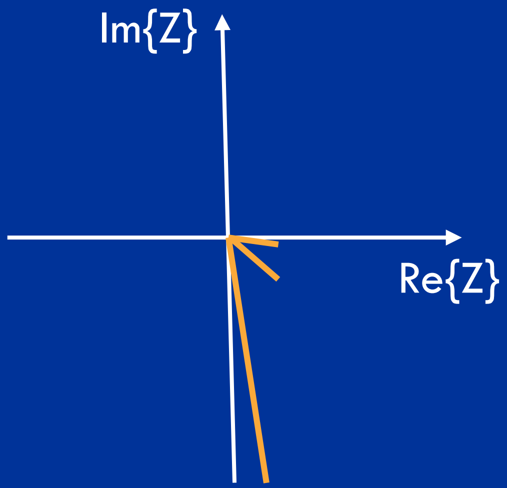


$$Z = 1E3 + (-j\,10E3) = \sqrt{(1E3)^2 + (1E4)^2}\angle \tan^{-1}\left(\frac{-1E4}{1E3}\right)$$
$$= \sqrt{1.01E8}\angle - 1.47\ rad$$
$$= \mathbf{10.05\ K\angle - 84.3^o}$$

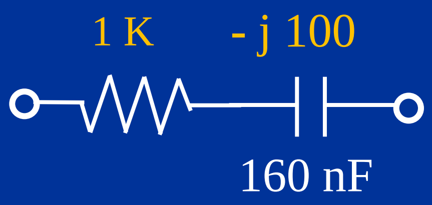
@ 1 KHz:



$$Z = \mathbf{1.414\ K\angle - 45^o}$$



@ 10 KHz:



$$Z = \mathbf{1.005\ K\angle - 5.7^o}$$

Complex Number Operations

Operations:

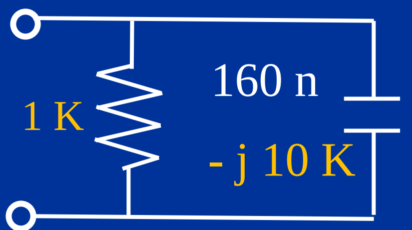
Let $z_1 = a_1 + j\, b_1 = r_1 \angle \theta_1$ and $z_2 = a_2 + j\, b_2 = r_2 \angle \theta_2$

Operation	***Best Representation***	***Result***
$z_3 = z_1 + z_2$	Rectangular	$z_3 = (a_1 + a_2) + j\,(b_1 + b_2)$
$z_3 = z_1 - z_2$	Rectangular	$z_3 = (a_1 - a_2) + j\,(b_1 - b_2)$
$z_3 = z_1\ z_2$	Polar	$z_3 = r_1\, r_2 \angle(\theta_1 + \theta_2)$
$z_3 = z_1 / z_2$	Polar	$z_3 = r_1 / r_2 \angle(\theta_1 - \theta_2)$
Conjugate	Either	$z^* = a - j\,b$ or $z^* = r \angle -\theta$

Problem Set 2

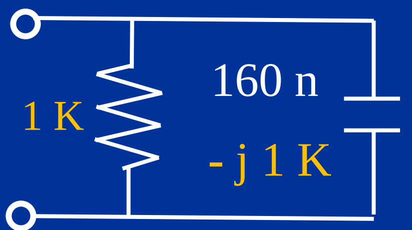
HINT: Recall $Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$

@ 100 Hz:



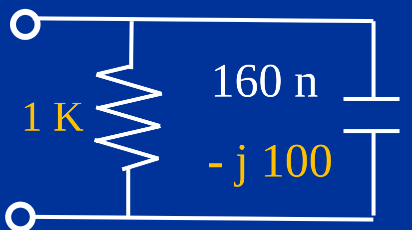
Z =

@ 1 KHz:



Z =

@ 10 KHz:



Z =

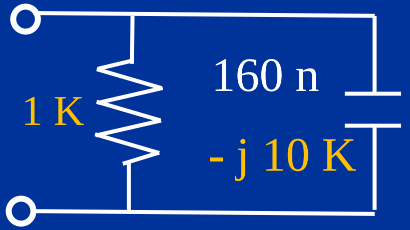
Problem Set 2 Solution

Operations:

Let $z_1 = a_1 + j\,b_1 = r_1\angle\theta_1$ and $z_2 = a_2 + j\,b_2 = r_2\angle\theta_2$

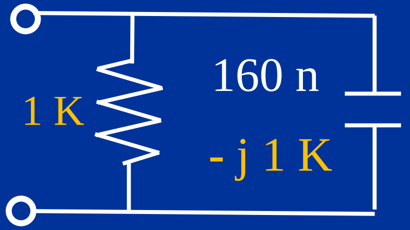
Operation	*Best Representation*	*Result*
$z_3 = z_1 + z_2$	Rectangular	$z_3 = (a_1+a_2) + j\,(b_1 + b_2)$
$z_3 = z_1 - z_2$	Rectangular	$z_3 = (a_1-a_2) + j\,(b_1 - b_2)$
$z_3 = z_1\ z_2$	Polar	$z_3 = r_1\, r_2\ \angle(\theta_1 + \theta_2)$
$z_3 = z_1 / z_2$	Polar	$z_3 = r_1 / r_2\ \angle(\theta_1 - \theta_2)$
Conjugate	Either	$z^* = a - j\,b$ *or* $z^* = r\ \angle -\theta$

@ 100 Hz:



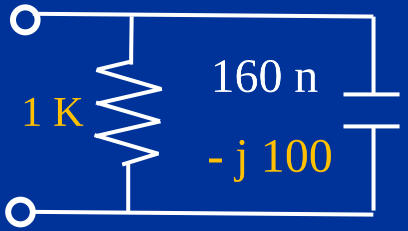
$$Z = \frac{(1E3)(-j\ 10E3)}{1E3\ \ -j\ 10E3} = \frac{(1E7\angle -90^o)}{10.05E3\ \angle -84.3^o} = 995\ \angle -5.71^o$$

@ 1 KHz:



$$Z = \ 707.1\ \angle -45^o$$

@ 10 KHz:



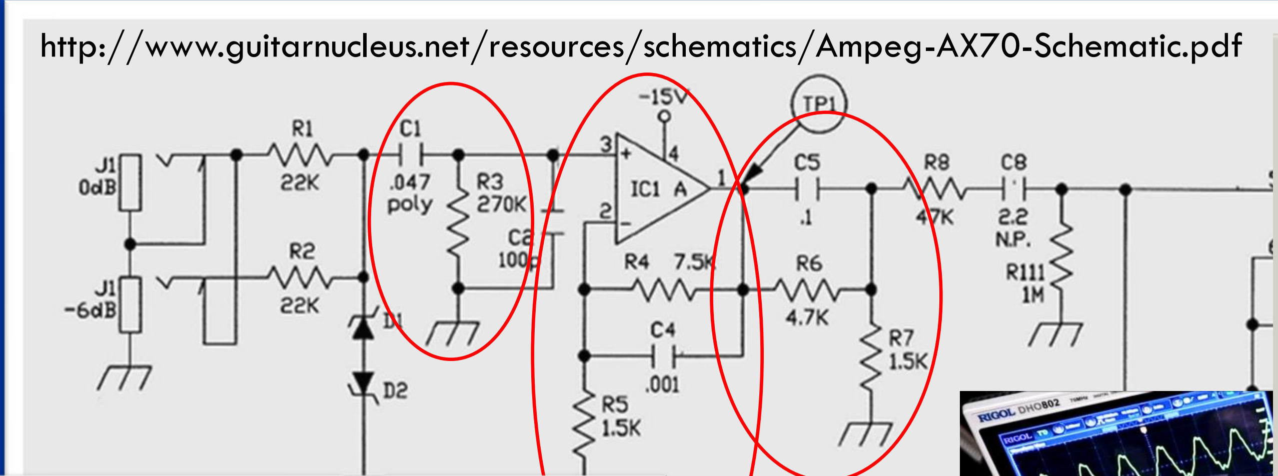
$$Z = \ 99.5\ \angle -84.3^o$$

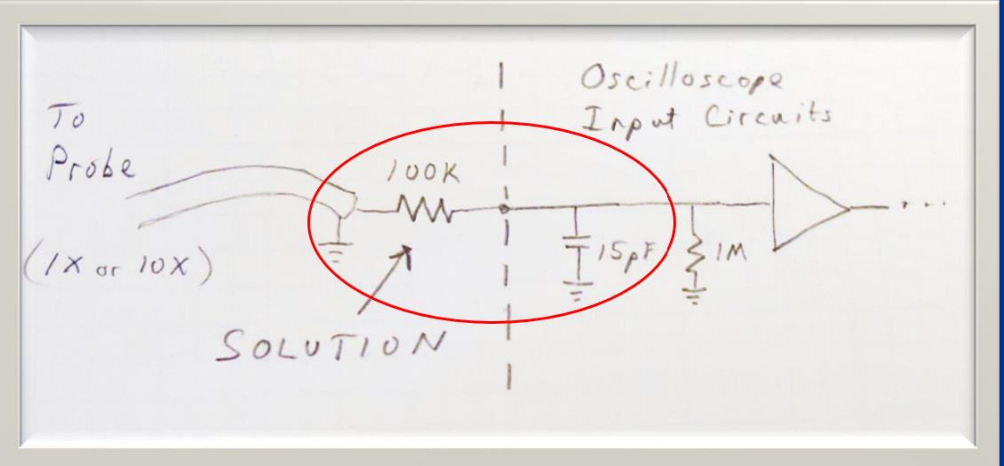
Today's Topics

- Complex Numbers and Operations
- Phasors, Impedance, and Circuit Analysis
- Plotting Frequency Response

Remember the Goal …

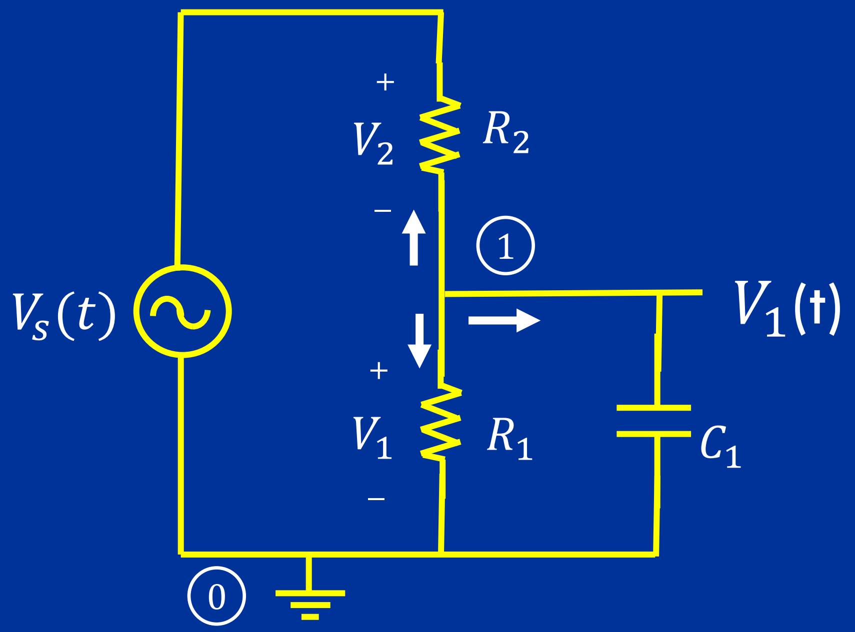
Analyze, understand, and design circuits !





Brute Force Approach

KCL-based "Nodal Analysis" at node 1 : $\sum I_{leaving-node-1} = 0$



$$\boldsymbol{(V_1(t) - V_s(t))\frac{1}{R_2} + V_1(t)\left(\frac{1}{R_1}\right) + C_1\frac{dV_1(t)}{dt} = 0}$$

Solve using differential equations and matrix algebra techniques

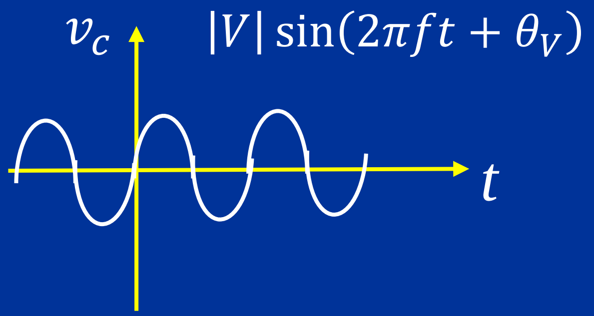
Painful for large circuits

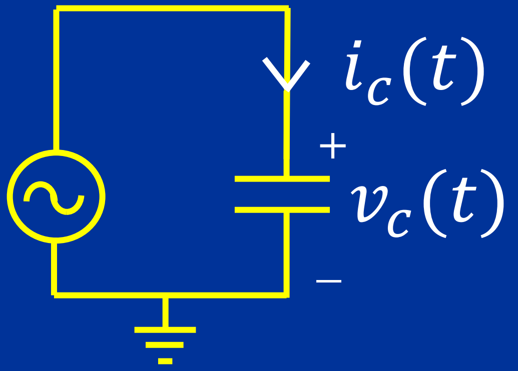
Use "Phasors" and "Impedances" instead !

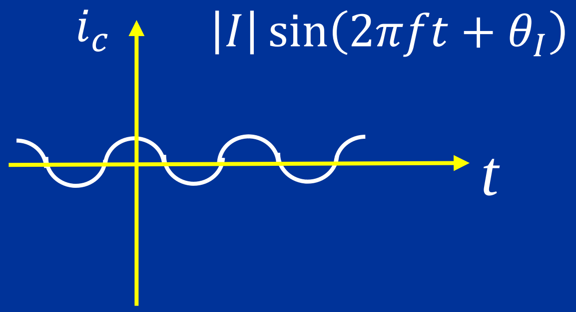
Phasors

Complex numbers represent the Magnitudes *and* Phases of sinewave voltages or currents in a circuit

In "real" world:





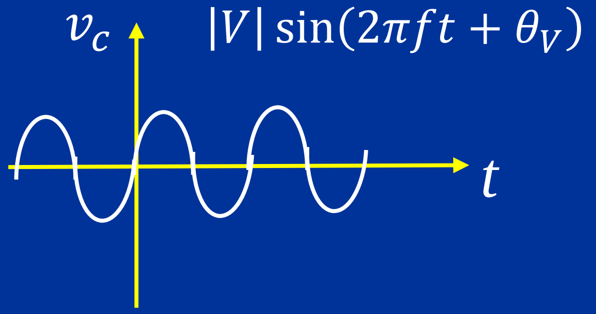


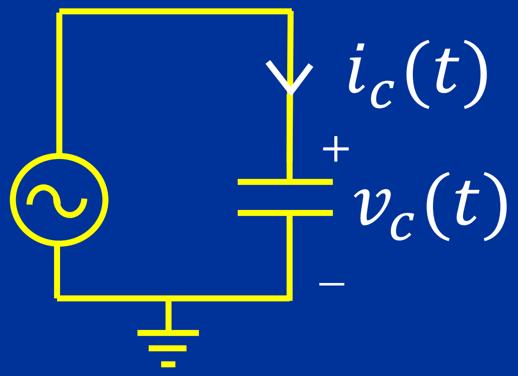
Phasors:

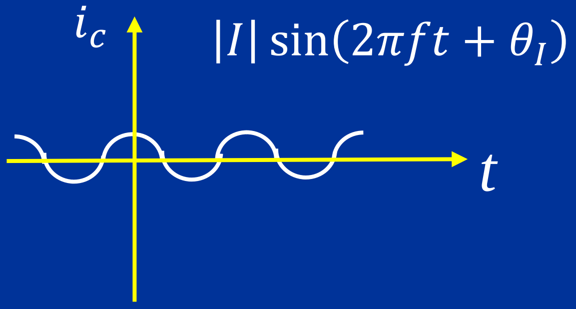
$$V = |V|\angle\theta_V = V_{real} + j\,V_{imag}$$

$$I = |I|\angle\theta_I = I_{real} + j\,I_{imag}$$

Reactance X relates *V* and *I* *Magnitudes*







From Physics: $i_c(t) = C\dfrac{d\,v_c(t)}{dt} = C|V|\,2\pi f\cos(2\pi ft + \theta_V) = \underbrace{|V|\,2\pi fC}_{|I|}\sin(2\pi ft + \underbrace{\theta_V + 90^o}_{\theta_I})$

Hence: $|I| = |V|\,2\pi fC$ *or* $|V| = |I|\underbrace{\dfrac{1}{2\pi fC}}_{X_c}$

Impedance Z relates *V* and *I* *Phasors*

Recall:

$$i_c(t) = C\frac{d\,v_c(t)}{dt} = C|V|\,2\pi f\,\cos(2\pi f t + \theta_V)$$

$$= \underbrace{|V|\,2\pi f C}_{|I|}\ \sin(2\pi f t + \underbrace{\theta_V + 90^o}_{\theta_I})$$

$$|V| = |I|\underbrace{\left(\frac{1}{2\pi f C}\right)}_{X_c}$$

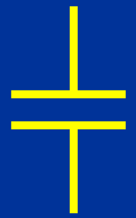
To capture effect on phase also:

$$V = I\,X_c\,\underbrace{(-j)}_{1\angle-90^o} = I\underbrace{\left(\frac{1}{j\,2\pi f C}\right)}_{Z_c} = I\,(0 - j\,X_c)$$

Resistance is zero in ideal cap

Impedance is purely "imaginary"

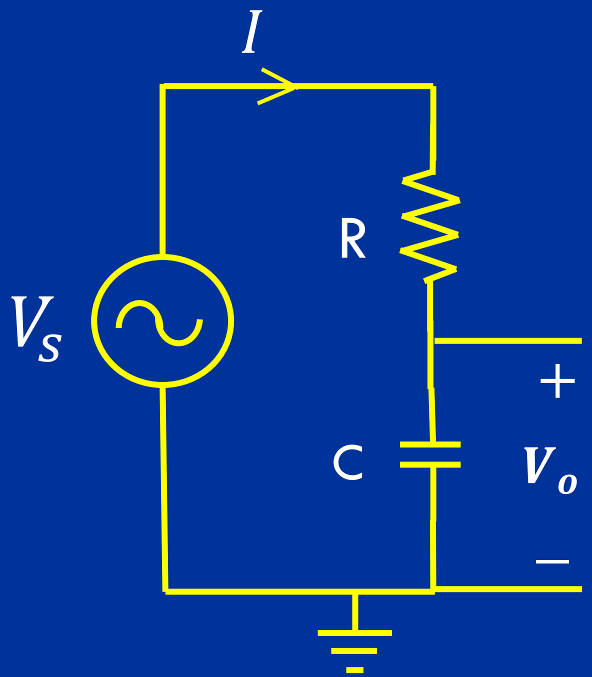
Circuit Analysis Example



$$Z_c = \frac{1}{j\,2\pi f C} = \frac{1}{j\,\omega\,C}$$

Frequency in Hz

Frequency in rad/sec



$$\boldsymbol{V_o} = \frac{Z_c}{(R + Z_c)}\boldsymbol{V_s}$$

$$= \frac{\frac{1}{j\omega C}}{\left(R + \frac{1}{j\omega C}\right)}\boldsymbol{V_s}$$

$$= \frac{1}{(1 + j\omega RC)}\boldsymbol{V_s}$$

$$= \frac{1}{\left(1 + j\frac{\omega}{\omega_o}\right)}\boldsymbol{V_s} \quad \text{where} \quad \omega_o = \frac{1}{RC}$$

Or, $\dfrac{V_o}{V_s} = \dfrac{1}{\left(1 + j\frac{f}{f_o}\right)}$ where $f_o = \dfrac{1}{2\pi RC}$

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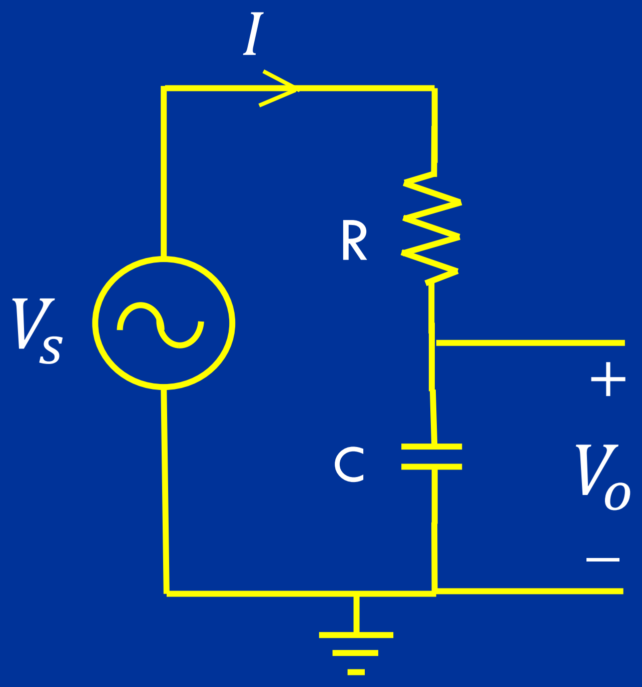
Frequency Response Plot(s)

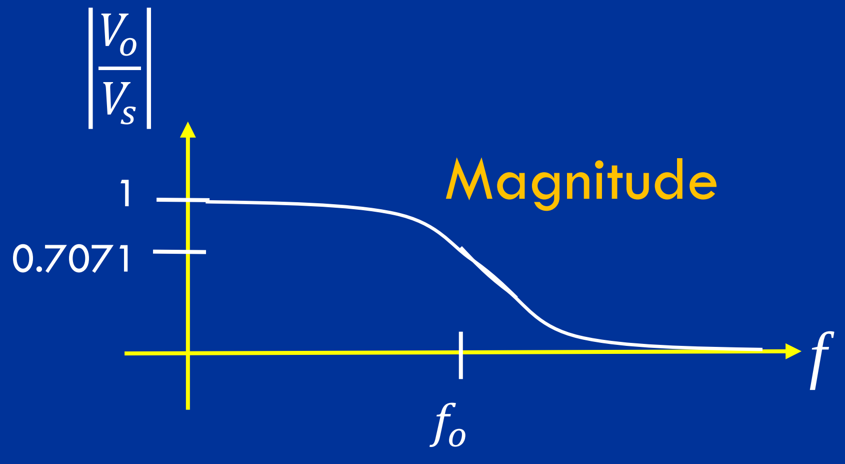


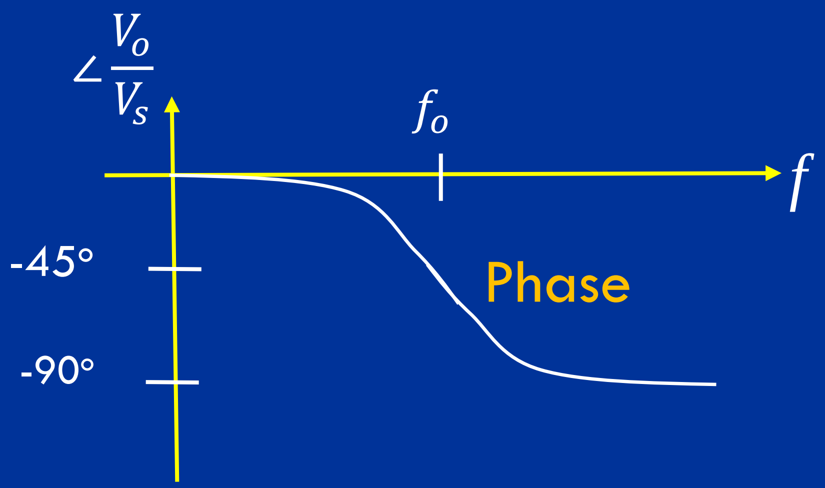
"Transfer Function" or "Gain"

$$H(f) = \frac{V_o(f)}{V_s(f)}$$

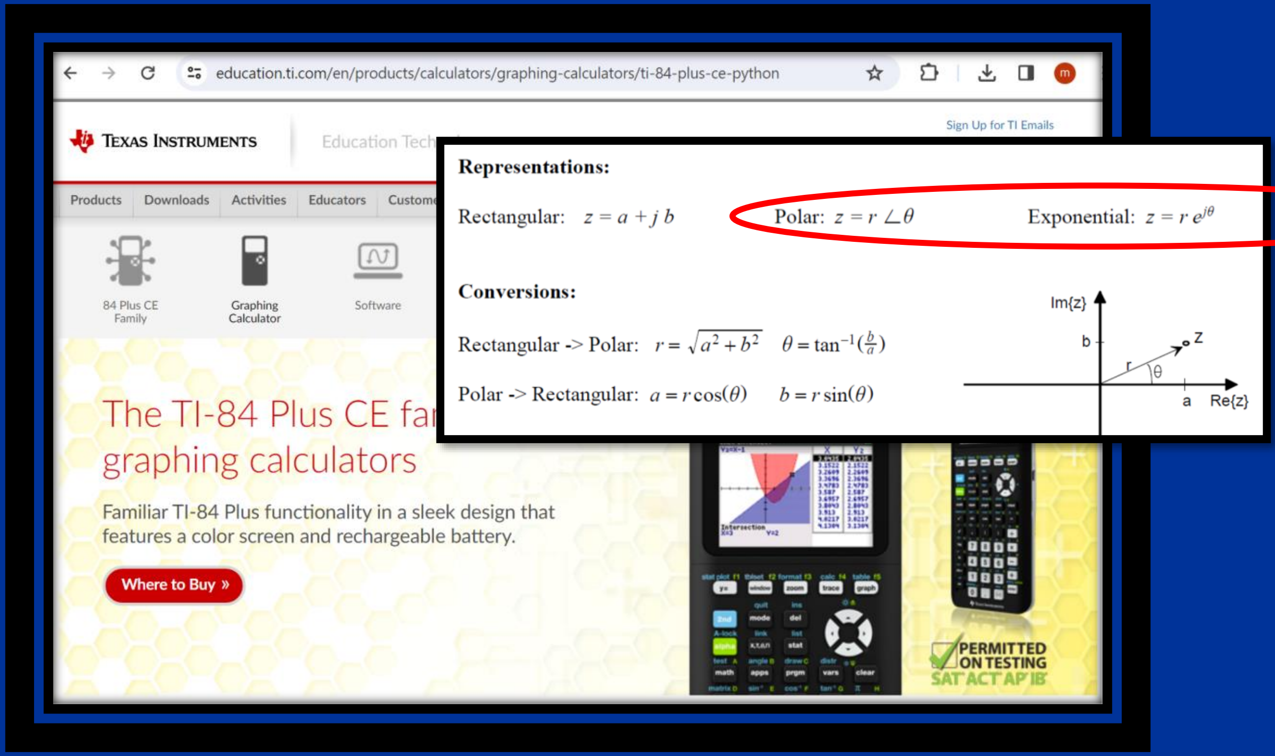
$$= \frac{1}{\left(1 + j\,\dfrac{f}{f_o}\right)}$$

NOTE: f_o *is often called* f_c
for "corner" or "cutoff" frequency.
(a.k.a. half-power or -3dB frequency)





(Algebraic) Complex Number Calculators



https://education.ti.com/en/products/calculators/graphing-calculators/ti-84-plus-ce-python

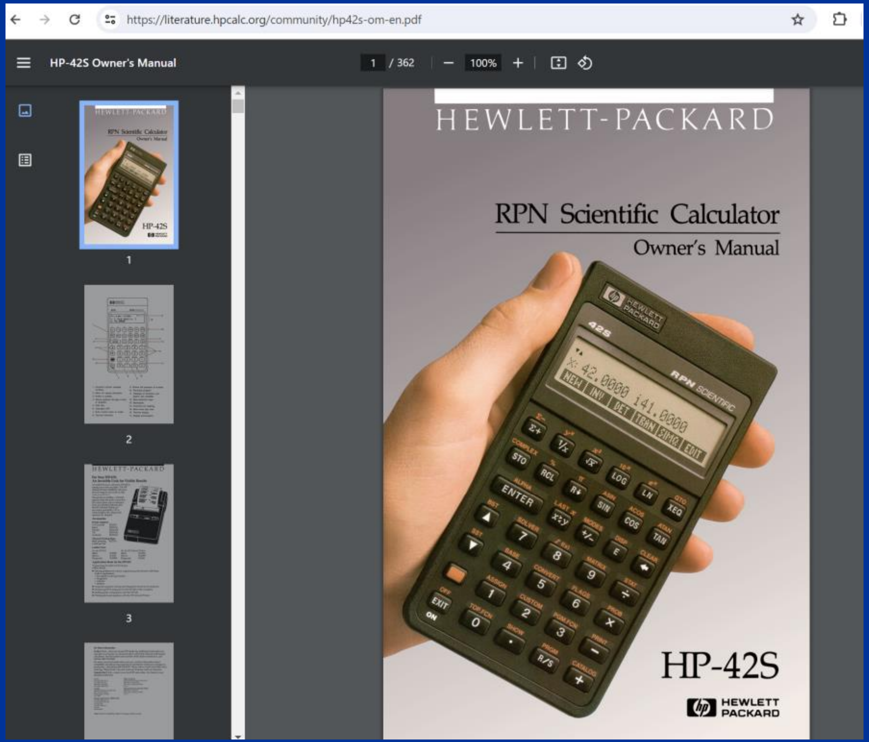
Calc84 (Cellphone app)

https://play.google.com/store/apps/details?id=scientific.graphing.calculator.t84.t36.t83&hl=en_US&gl=US

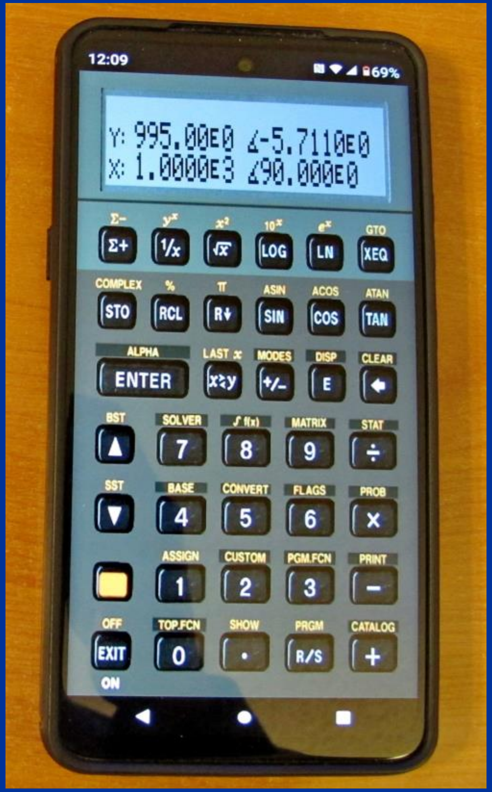
(RPN) Complex Number Calculators

Free 42s for Cellphone !

The original HP42S

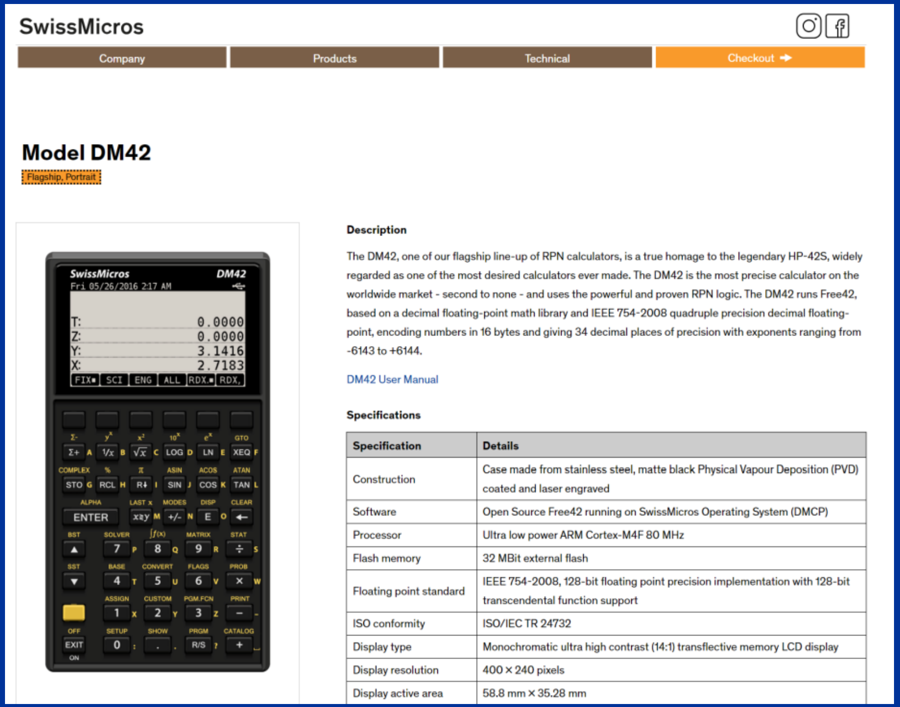


https://literature.hpcalc.org/community/hp42s-om-en.pdf



https://thomasokken.com/free42/

Re-creation by Swiss Micros



Description

The DM42, one of our flagship line-up of RPN calculators, is a true homage to the legendary HP-42S, widely regarded as one of the most desired calculators ever made. The DM42 is the most precise calculator on the worldwide market - second to none - and uses the powerful and proven RPN logic. The DM42 runs Free42, based on a decimal floating-point math library and IEEE 754-2008 quadruple precision decimal floating-point, encoding numbers in 16 bytes and giving 34 decimal places of precision with exponents ranging from -6143 to +6144.

DM42 User Manual

Specifications

Specification	Details
Construction	Case made from stainless steel, matte black Physical Vapour Deposition (PVD) coated and laser engraved
Software	Open Source Free42 running on SwissMicros Operating System (DMCP)
Processor	Ultra low power ARM Cortex-M4F 80 MHz
Flash memory	32 MBit external flash
Floating point standard	IEEE 754-2008, 128-bit floating point precision implementation with 128-bit transcendental function support
ISO conformity	ISO/IEC TR 24732
Display type	Monochromatic ultra high contrast (14:1) transflective memory LCD display
Display resolution	400 × 240 pixels
Display active area	58.8 mm × 35.28 mm

https://www.swissmicros.com/products

Thanks for Watching !

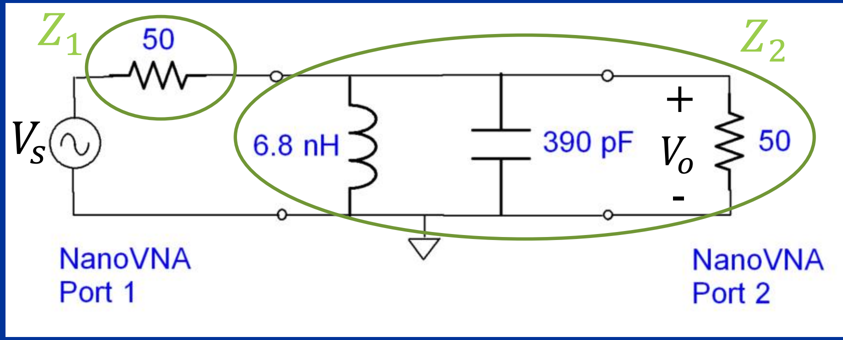
Homework Problems

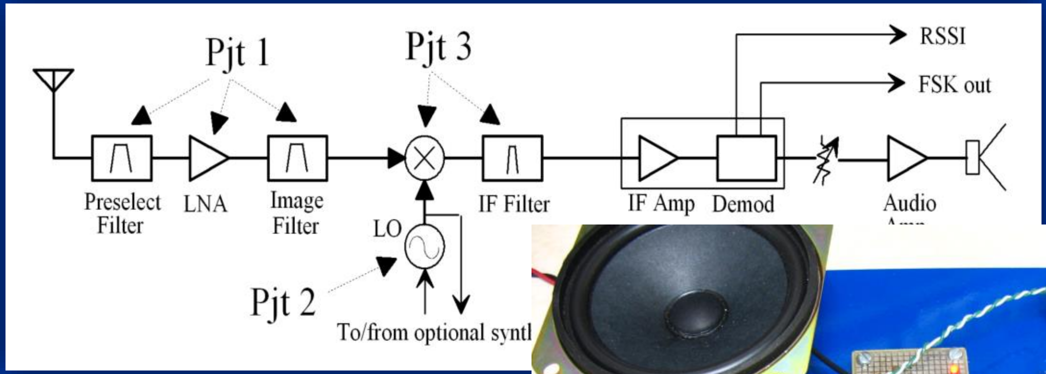
☺

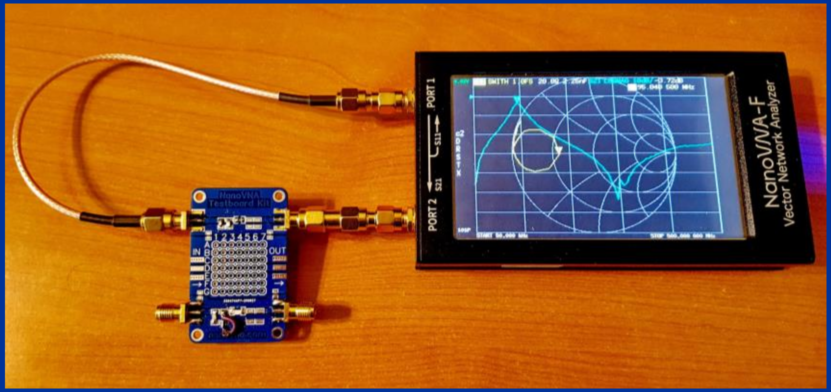
Application: Radio/Wireless

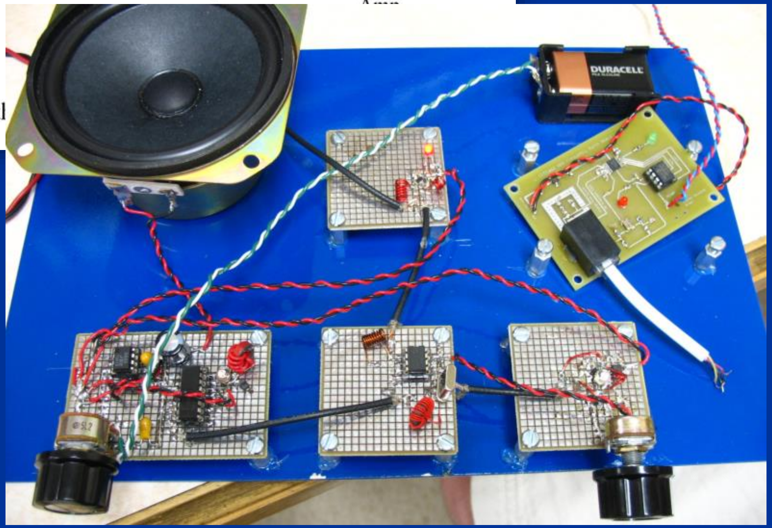
From Radio Design 101

Episode 1- Transceivers and (Bandpass) Filters







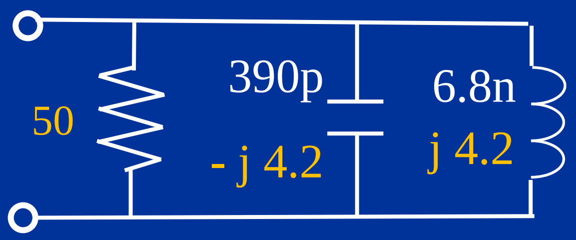


Homework 6.1 (Resonant Circuits)

See Radio Design 101, Episode 1

HINT: Do pairwise reduction for quicker solution, with more insight

@ 98 MHz:



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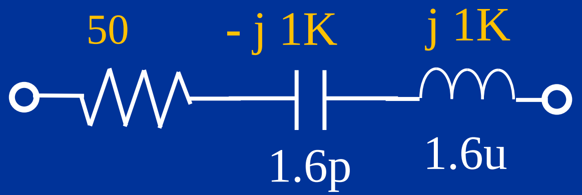
@ 98 MHz:

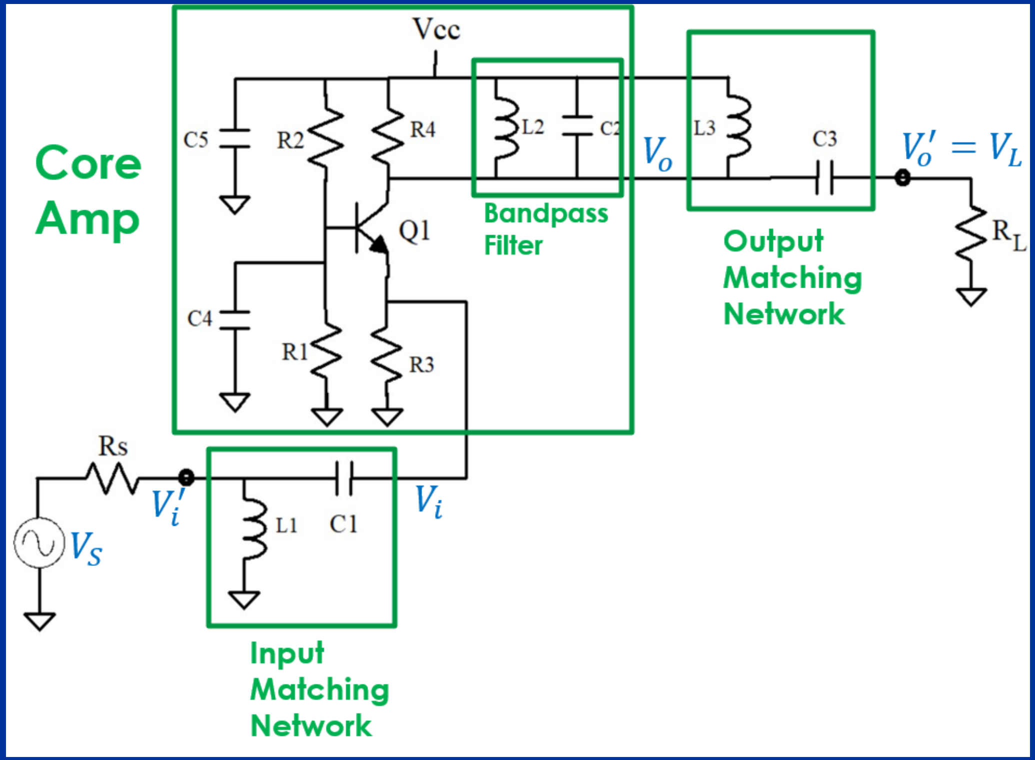


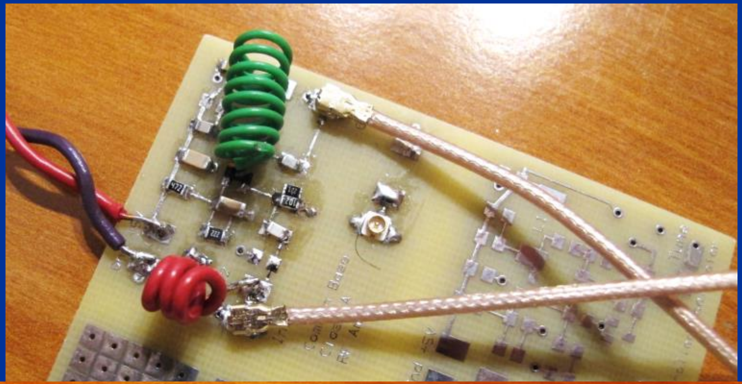
Z =

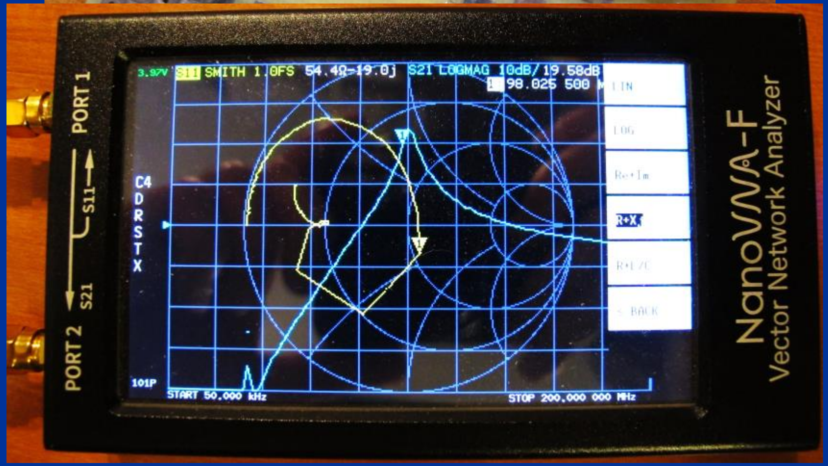
Application: Radio/Wireless

From Radio Design 101

Episode 2- Impedance Matching





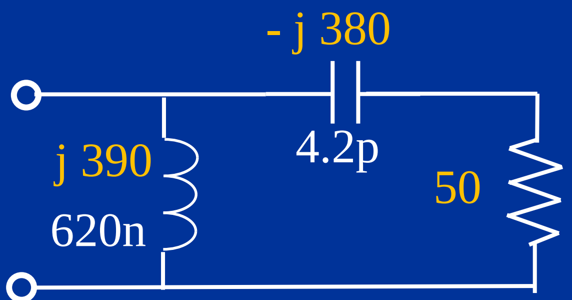


Homework 6.2 (Matching Networks)

See Radio Design 101, Episode 2

HINT: Do pairwise reduction for quicker solution, with more insight …

@ 100 MHz:



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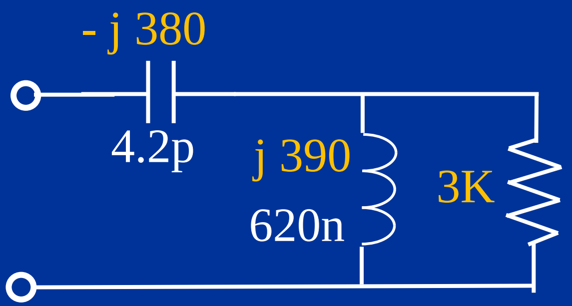
@ 100 MHz:



Z =